PRACTICE FINAL (AGOL) - SOLUTIONS

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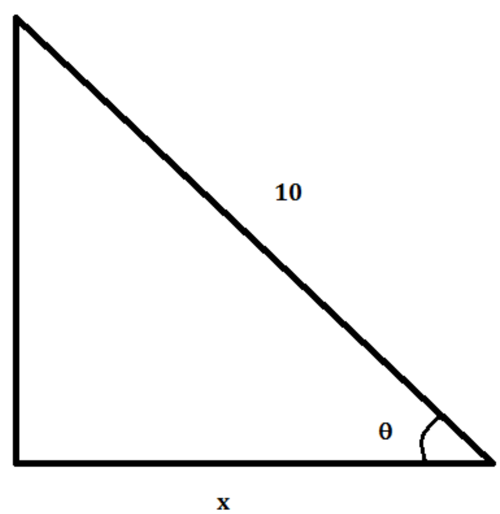
1. 1) Picture:

1A/Practice Exams/Ladder.png



2) We want to find $\frac{d\theta}{dt}$ when $x = 6$
3) We know $\cos(\theta) = \frac{x}{10}$
4) Differentiating, we get: $-\sin(\theta)\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$.
5) However, $\frac{dx}{dt} = 2$, and if $x = 6$, then we get our usual $6 - 8 - 10$-triangle, so $\sin(\theta) = \frac{8}{10} = \frac{4}{5}$. Putting everything together, we get $-\frac{8}{10}\frac{d\theta}{dt} = \frac{6}{10}$, so $\frac{d\theta}{dt} = \frac{6}{8} = \boxed{\frac{3}{4} \text{ ft/s}}$

2. **Note:** I think the problem should say $x > 1$

Define $\boxed{f(x) = x\ln(x) - (x-1)}$, by the Mean Value Theorem applied to f at $(1, x)$, we get that there is a $c \in (1, x)$ such that $\frac{f(x)-f(1)}{x-1} = f'(c)$. But $f(1) = 0$, and $f'(c) = \ln(c) + 1 - 1 = \ln(c) > 0$ (since $c > 1$), so we get: $\frac{f(x)}{x-1} > 0$, so multiplying by $x - 1 > 0$, we get $f(x) > 0$, so $x\ln(x) - (x-1) > 0$, so

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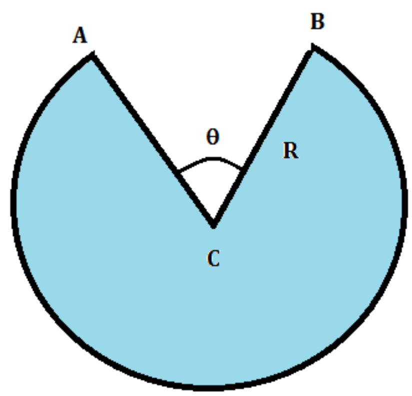
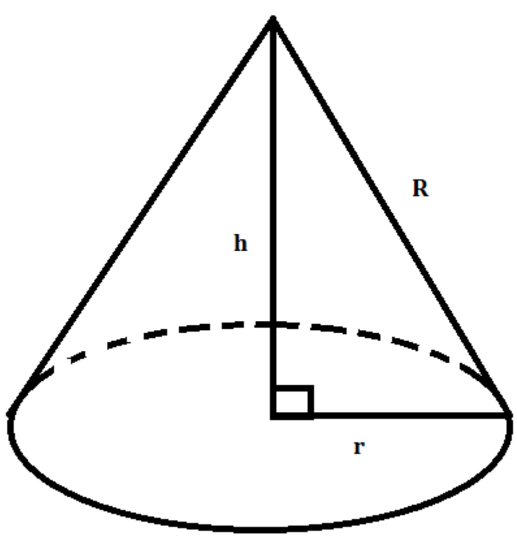
$\boxed{x \ln(x) > x - 1}$.

3. 1) The circular piece of paper looks as follows:
1A/Practice Exams/Drinking cup.png



And when you join the edges, your drinking cup should look as follows:
1A/Practice Exams/Drinking cup 2.png



2) You want to maximize the volume $V = \frac{\pi}{3}r^2h$, where r is the radius of the cup, and h is the height. But by the Pythagorean theorem, you know that $r^2 + h^2 = R^2$, so $r^2 = R^2 - h^2$, so $\boxed{V(h) = \frac{\pi}{3}(R^2 - h^2)h}$

3) The only constraint is $0 \leq h \leq R$ ($h \geq 0$ is clear, and $h \leq R$ is because we want the side of the triangle to be smaller than its hypothenuse)

4) $V'(h) = \frac{\pi}{3}(-2h)h + \frac{\pi}{3}(R^2 - h^2) = -\pi h^2 + \frac{\pi}{3}R^2 = 0 \Leftrightarrow h^2 = \frac{R^2}{3} \Leftrightarrow h = \frac{R}{\sqrt{3}}$. Now $V(0) = V(R) = 0$, and $V\left(\frac{R}{\sqrt{3}}\right) = \frac{2\pi R^3}{9\sqrt{3}} > 0$. Hence, by the closed interval method, the maximum volume is $\boxed{\frac{2\pi R^3}{9\sqrt{3}}}$

4. Let F be an antiderivative of $\ln(x)$. Then $\int_{x^2}^{1+x^2} \ln(t)dt = F(1+x^2) - F(x^2)$, so:

$$\frac{d}{dx}\int_{x^2}^{1+x^2} \ln(t)dt = F'(1+x^2)(2x) - F'(x^2)(2x) = \ln(1+x^2)(2x) - \ln(x^2)(2x)$$

5. The slope of the tangent line to $x = y^3$ is $\frac{dy}{dx}$, where $1 = 3y^2\frac{dy}{dx}$, so $\boxed{\frac{dy}{dx} = \frac{1}{3y^2}}$. And the slope of the tangent line to $y^2 + 3x^2 = 5$ is $\frac{dy}{dx}$, where $2y\frac{dy}{dx} + 6x = 0$, so $\boxed{\frac{dy}{dx} = -\frac{3x}{y}}$. Now when the two curves intersect, the slope of the tangent line to the first curve, $\frac{dy}{dx} = \frac{1}{3y^2} = \frac{1}{3\frac{x}{y}} = \frac{y}{3x}$ (here, I used the fact that $x = y^3$, so dividing by y on both sides, you get $\frac{x}{y} = y^2$), is the negative reciprocal of the slope of the tangent line to the second curve, $\frac{dy}{dx} = -\frac{3x}{y}$! Hence the tangent lines are perpendicular when the two curves intersect!

6. (a) First of all, using the substitution $u = \tan^{-1}(x)$, with $du = \frac{1}{1+x^2}dx$, we get $\int_0^1 \frac{\tan^{-1}(x)}{1+x^2}dx = \int_0^{\frac{\pi}{4}} udu = \left[\frac{u^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi^2}{32}$. Hence:

$$\int_0^1 x - \frac{\tan^{-1}(x)}{1+x^2}dx = \left[\frac{x^2}{2}\right]_0^1 - \frac{\pi^2}{32} = \frac{1}{2} - \frac{\pi^2}{32}$$

(b) $\int_0^2 \sqrt{4-x^2}dx = \frac{\pi(2)^2}{4} = \boxed{\pi}$ (because the integral is the area of the quarter of circle of radius 2)

(c) Let $u = 1 + e^x$, then $du = e^x dx$, so:

$$\int e^x\sqrt{1+e^x}dx = \int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(1+e^x)^{\frac{3}{2}} + C$$

7. (a) Domain: $x \neq 0$
   (b) Intercepts: None
   (c) Symmetry: None
   (d) Asymptotes: $y = 0$ is a H.A. at $-\infty$ (by l'Hopital's rule) , $x = 0$ is a V.A. (since $\lim_{x\to 0^+} \frac{e^x}{x} = \infty$ and $\lim_{x\to 0^+} \frac{e^x}{x} = -\infty$ by direct computation)
   (e) Intervals of increase or decrease: $f'(x) = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$. f is decreasing on $(-\infty, 0) \cup (0,1)$ and increasing on $(1,\infty)$
   (f) Local maximum and minimum values: $(1, e)$ is a local minimum by the first derivative test!
   (g) Concavity and points of inflection:

$$f''(x) = \frac{(e^x(x-1)+e^x)x^2 - e^x(x-1)(2x)}{x^4} = \frac{x^3e^x - 2x^2e^x + 2xe^x}{x^4} = \frac{e^x(x^2-2x+2)}{x^3}$$

   But notice that $x^2 - 2x + 1 = (x-1)^2 + 1 > 0$, so the sign of f'' only depends on the sign of x^3. In particular, f is concave down on $(-\infty, 0)$ and concave up on $(0,\infty)$. There are no inflection points!
   (h) Again, whip out your calculator and look at your rough sketch!

8 (a) $f'(x) = 1 - \frac{1}{\sqrt{x}} > 0$ when $x > 1$, so f is increasing when $x > 1$.

   (b) Let $y = x - 2\sqrt{x}$, all we need to do is to solve for x in terms of y. The trick is: Let $X = \sqrt{x}$, then $y = X^2 - 2X$, so $X^2 - 2X - y = 0$, so $(X-1)^2 + 1 - y = 0$, so $(X-1)^2 = y - 1$, so $X - 1 = \sqrt{y-1}$ (and this is legitimate, because since f is increasing when $x > 1$ by (a), we have $y = f(x) \geq f(1) = 1$), so $X = 1 + \sqrt{y-1}$, so $\sqrt{x} = 1 + \sqrt{y-1}$, so $x = \left(1+\sqrt{y-1}\right)^2$. Hence $\boxed{f^{-1}(x) = \left(1+\sqrt{x-1}\right)^2}$

   (c) We need to show that for all $M > 0$ there exists an N large enough such that when $x > N$, then $f(x) > M$.
   Let $M > 0$ be given, we want $x - 2\sqrt{x} > M$. But notice $x - 2\sqrt{x} = \left(\sqrt{x}\right)^2 - 2\sqrt{x} + 1 - 1 = \left(\sqrt{x}-1\right)^2 - 1 > M$, which gives $\left(\sqrt{x}-1\right)^2 > M+1$, hence $\sqrt{x} - 1 > \sqrt{M+1}$, so $x > (1+\sqrt{M+1})^2$.
   Now choose $N = (1+\sqrt{M+1})^2$, then if $x > N$, $x - 2\sqrt{x} > x > N = M$, and we're done!

9. The picture is on page 437!

   Shell method: $K = 0$, $|x| = x$, Outer = $2\sqrt{R^2 - x^2}$ (use the fact that $x^2 + y^2 = R^2$), Inner = 0,

$$\int_r^R 2\pi x(2\sqrt{R^2 - x^2})dx = \frac{4\pi}{3}(R^2 - r^2)^{\frac{3}{2}} = \frac{4\pi}{3}\left(\left(\frac{h}{2}\right)^2\right)^{\frac{3}{2}} = \frac{4\pi}{3}\frac{h^3}{8} = \frac{\pi h^3}{6}$$

(use the substitution $u = R^2 - r^2$, and the fact that $r^2 + (\frac{h}{2})^2 = R^2$ by the Pythagorean theorem)